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CONTROLLER SYNTHESIS FOR A STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY A LAPLACE EQUATION

Research article

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Abstract

This article is concerned with an optimal control problem governed by a Laplace equation. Initially, the optimal control problem, governed by a system of partial differential elliptic equations of the second order, is considered. The case of a system, that is singular according to Lions, is considered. In this system a given control may give rise to either no of any state or, on the contrary, the infinitely many ones or that to a single but unstable state [1]. In this situation, the application of the classic optimal control theory is either very difficult or impossible. Special methods applicable to the control problems, governed by singular distributed systems, are developed in the works of Zh. L. Lions, I. Ekland, P. Marselini, G. Mossino, P. Rivera, and of many other authors. But it should be noted that in most of these works the simplest problem statement is discussed. It is defined by the fact that the set of admissible processes, i.e., the processes, among which we seek the minimum of certain functional, is described by a differential equation and the connected with it, boundary conditions only. In the present work a more general and complex case is considered, namely, the case that in the description of the above-mentioned set there are so-called state constraints. This implies that the phase vector of a system does not leave the given set. In such a statement the optimal control problem, governed by a distributed singular system, is, undoubtedly, of substantial interest. Next it will be shown that the optimal process in this problem is generated by a nonlinear optimal controller and its equation will be obtained.

Keywords: maximum principle of Pontryagin's type, Laplace equation, optimal controller.

СИНТЕЗ РЕГУЛЯТОРА В ЗАДАЧЕ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ УРАВНЕНИЕМ ЛАПЛАСА С ФАЗОВЫМИ ОГРАНИЧЕНИЯМИ

Научная статья

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Аннотация

Статья посвящена задаче оптимального управления объектом, поведение которого описывается уравнением Лапласа. В начале рассматривается задача оптимального управления системой дифференциальных уравнений в частных производных эллиптического типа второго порядка. Исследован случай так называемой сингулярной системы [1]. В такой системе заданному управлению может не соответствовать никакое состояние управляемого объекта, либо, напротив, таких состояний может быть бесконечно много, либо состояние может быть только одно, но неустойчивое. В такой ситуации применение классической теории оптимального управления оказывается либо очень затруднительным, либо вообще невозможным. Специальные методы, применимые к задачам управления сингулярными распределенными системами были развиты в работах Ж.Л. Лионса, И. Экланда, П. Марселини, Ж. Моссино, П. Ривера и многих других авторов. Однако стоит заметить, что в подавляющем большинстве этих работ рассматривается простейшая постановка задачи. Она характеризуется тем, что множество допустимых процессов, то есть процессов, среди которых ищется минимум некоторого функционала, описывается только дифференциальным уравнением и связанными с ним граничными условиями. В настоящей работе рассмотрен более общий и сложный случай, а именно, случай, когда в описании упомянутого множества присутствуют так называемые фазовые ограничения. Они требуют, чтобы фазовый вектор системы не покидал заданного множества. В такой постановке задача оптимального управления сингулярной распределенной системой, несомненно, представляет значительный интерес. В статье показано, что оптимальный процесс в данной задаче порождается нелинейным оптимальным регулятором и получено его уравнение.

Ключевые слова: принцип максимума Понтрягина, уравнение Лапласа, оптимальный регулятор.

Introduction

Recent decades have witnessed a sustained interest to the maximum principle of Pontryagin's type for optimal control problems governed by partial differential equations; see e.g., [3], [4], [5] and the literature therein. The introducing of state constraints into the problem formulation has earned the additional difficulties and increasing complexity, whereas such constraints are relevant to many applications [6], [7], [8]. The optimal control theory of PDE's offers an extremely rich variety of problems. Among them, there are optimal control problems with state constraints for plants described by Laplace equation.

In this article, the optimal control problem governed by a Laplace equation is considered. This control system is the singular, according to Lions [1].

The optimal control problem governed by a system of elliptic equations. The case of phase constraints

Let Ω be an open bounded subset of $\mathbb{R}^l$ with the Lipshitz boundary Γ, $K \subset \mathbb{R}^m$ be a nonempty set, and $f : \Omega \times \mathbb{R}^h \times \overline{K} \to \mathbb{R}^h$. Consider the following system:

$$\begin{cases} Ay = f[x, y(x), u(x)], u(x) \in K, x \in \Omega \\ y|_\Gamma = 0 \end{cases} \quad (1)$$

where $y(\cdot) = (y_{(k)}(\cdot))_{k=1}^h : \Omega \to \mathbb{R}^h$ *is a state, u(·) is a control*, and A is an elliptic differential operator of the second order: Ay(·)=p(·), where $p(x) = |p_k(x)|_{k=1}^h$, $p_k(x) = -\sum_{i,j=1}^l \frac{\partial}{\partial x_j}[a_{ij}^{(k)}(x)\frac{\partial y_k}{\partial x_i}] \forall k = 1, \ldots, h, a_{ij}^{(k)} \in C^{0,1}(\overline{\Omega})$ and $\sum_{i,j=1}^l a_{ij}^{(k)} \xi_i \xi_j \geq \Lambda|\xi|^2 \forall \xi \in \mathbb{R}^l, x \in \bar{\Omega}, k = 1, \ldots, h$ for a certain Λ>0.

Here $C^{0,\alpha}(\overline{\Omega}), \alpha \in (0, 1]$ is a space of all continuous in $\overline{\Omega}$ functions satisfying the Hoelder condition: $\sup_{x_1, x_2 \in \Omega} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} < \infty$. We consider controls $u(\cdot) \in L^\infty(\Omega \to K)$ and will seek solutions of problem (1) in a class $H_0^1(\overline{\Omega} \to \mathbb{R}^h)$. Recall that $H_0^1(\overline{\Omega} \to \mathbb{R}^h)$ is a closure of set $C_0^\infty(\bar{\Omega} \to \mathbb{R}^h) := \{\varphi(\cdot) \in C^\infty(\Omega \to \mathbb{R}^h) : \varphi(x) = 0$ for *x∉M, where M⊂int Ω* is compact set} in $H^1(\bar{\Omega} \to \mathbb{R}^h) := \left\{\varphi(\cdot) \in L^2(\bar{\Omega} \to \mathbb{R}^h) : \frac{\partial \varphi(\cdot)}{\partial x_i} \in L^2(\bar{\Omega} \to \mathbb{R}^h), i = 1, \ldots, l\right\}$. The norm in $H^1(\overline{\Omega} \to \mathbb{R}^h)$ is defined by the equality $|\varphi(\cdot)|_{H^1}^2 := |\varphi(\cdot)|_2^2 + \sum_{i=1}^l |\partial\varphi(\cdot)/\partial x_i|_2^2$. Suppose that functions $L : \Omega \times \mathbb{R}^h \times \overline{K} \to \mathbb{R}$ and $g(\cdot) = |g_i(\cdot)|_{i=1}^q : \overline{\Omega} \times \mathbb{R}^h \to \mathbb{R}^q$ are given. Consider an optimal control problem:

$$J(y, u) := \int_\Omega L[x, y(x), u(x)]dx \to inf \quad (2)$$

on the set $D := \{[y(\cdot), u(\cdot)] : y(\cdot) \in H_0^1(\bar{\Omega} \to \mathbb{R}^h) \cap C^{0,\alpha}(\bar{\Omega} \to \mathbb{R}^h), u(\cdot) \in L^\infty(\Omega \to \mathbb{R}^m)$ is valid and $g_i(x, y(x)) \leq 0 \forall x \in \Omega, i = 1, \ldots, q\}$

We also assume the following:

1. For almost all $x \in \Omega$ a function *L(x,y,u)* is continuous in *(y,u)* together with derivative *∂L/∂y*. For all *(y,u)* the function *L(x,y,u)* is measurable by *x* and for any *r>0* for a certain $\alpha_r(\cdot) \in L(\Omega \to \mathbb{R})$ an estimate $|L(x,y,u)|+|(\partial L(x,y,u))/\partial y| \leq \alpha_r(x)$ holds for almost all *x∈Ω* and all *y,u∈K* such that *|y|≤r, |u|≤r*.

2. For almost all *x∈Ω* a function *f(x,y,u)* is continuous in *(y,u)* together with derivative *∂f/∂y*. For any *(y,u)* a function *f(x,y,u)* is measurable by *x*. There exists *s>l/(l-1)* such that for any *r* for a certain $\beta_r(\cdot) \in L^s(\Omega \to \mathbb{R})$ the estimate $|f(x,y,u)|+|(\partial f(x,y,u))/\partial y| \leq \beta_r(x)$ is valid for almost all *x∈Ω* and any y,u∈K such that *|y|≤r, |u|≤r*.

3. Functions $g_i(x,y)$ are continuous in (x,y) together with derivative $\partial g_i/\partial y$ and $g_i(x,0)<0$ $\forall x \in \Gamma$, *i=1,…,q*.

Denote by *M(Ω)* a space of all real regular Borel charges in *Ω*. It can be identified with the dual to $C^0(\Omega)$ space [9], where $C^0(\Omega)=\{\varphi(\cdot) \in C(\Omega) : \varphi(x)=0 \ \forall x \in \Gamma\}$. Denote by a symbol $W_0^{1,\sigma}(\Omega \to \mathbb{R}^h)$, $\sigma \in [1,\infty)$, a closure of space $C_0^\infty(\Omega \to \mathbb{R}^h)$ in $W^{1,\sigma}(\Omega \to \mathbb{R}^h) := \left\{\varphi(\cdot) \in L^2(\Omega \to \mathbb{R}^h) : \frac{\partial \varphi(\cdot)}{\partial x_i} \in L^\sigma(\Omega \to \mathbb{R}^h), i = 1, \ldots, l\right\}$. In $W_0^{1,\sigma}(\Omega \to \mathbb{R}^h)$ a norm $|\varphi(\cdot)| := (|\varphi(\cdot)|_2^2 + \sum_{i=1}^l |\partial\varphi(\cdot)/\partial x_i|_\sigma^\sigma)^{1/\sigma}$ is considered.

Theorem 1.

Let hypotheses 1-3 be satisfied and (y^0,u^0) be an optimal process in problem (2). Then there exists a function $\psi(\cdot) \in W_0^{1,\sigma}(\Omega \to \mathbb{R}^h)$, where $\sigma < l/(l-1)$, the charges $\mu_i(dx) \in M(\Omega), i=1,\ldots,q$ and a number $\lambda^0 \in R$ are such that

$$A^*\psi(x) - \nabla_y \mathrm{H}\left[\mathrm{x}, y^0(x), u^0(x)\right] + \sum_{i=1}^q \mu_i(dx) \nabla_y g_i\left[\mathrm{x}, y^0(x)\right] = 0, x \in \Omega \quad (3)$$

$$\mathrm{H}\left[\mathrm{x}, y^0(x), u^0(x)\right] = \max_{v \in K} \mathrm{H}\left[\mathrm{x}, y^0(x), \mathrm{v}\right] \text{ for almost all } x \in \Omega \quad (4)$$

$$\lambda^0 \geq 0, \mu_i(dx) \geq 0, \operatorname{supp} \mu_i(dx) \subset \left\{x : g_i\left[\mathrm{x}, y^0(x)\right] = 0\right\} \forall i = 1, \ldots, q \quad (5)$$

$$\lambda^0 + \int_\Omega |\psi(x)|dx + \sum_{i=1}^q \mu_i(\Omega) > 0 \quad (6)$$

Here $H[x,y,u]=\psi^*(x)f(x,y,u)-\lambda^0 L(x,y,u)$ is a Hamiltonian function and $A^*\psi(x)=p(x)$, where $p(x) = |p_k(x)|_{k=1}^h$, $p_k(x) = -\sum_{i,j=1}^l \frac{\partial}{\partial x_i}\left[a_{ij}^{(k)}(x)\frac{\partial \psi_k}{\partial x_j}\right] \forall k = 1, \ldots, h$.

In (3) all the addends are considered as generalized functions [10]. This equation is of the elliptic type of the second order with respect to *ψ(·)*. The inclusion $\psi(x) \in W_0^{1,\sigma}(\Omega \to \mathbb{R}^h)$ involves the validity of the homogeneous Dirichlet boundary condition $\psi|_\Gamma=0$. The equation (3) with measures $\mu_i(dx)$ was studied in [9]. Relation (1) implies that $y^0|_\Gamma=0$ and therefore condition 3 results in $g_i[x,y^0(x)]<0$ $\forall x \in \Gamma, i=1,\ldots,q$. By inclusion (5) we have supp $\mu_i(dx) \subset int\ \Omega$ $\forall i=1,\ldots,q$. The proof of Theorem 1 is given in [2].

The optimal control problem governed by a Laplace equation. The case of phase constraints

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be an open bounded subset. Consider the following optimal control problem:

$$\Delta y = uy^3 - ku^2, u_- \le u \le u_+, x \in \Omega \quad (7)$$

$$y|_\Gamma = 0, y(x) \le \varphi(x) \forall x \in \bar{\Omega}, \int_\Omega y(x)^2 dx \to \min \quad (8)$$

where $y=y(x)\in\mathbb{R}$ is a state, $u=u(x)\in\mathbb{R}$ is a control, Δ – Laplace operator, numbers $k>0, 0<u\_-<u\_+$ and a continuous function $\varphi(x)$, $x \in \overline{\Omega}$ are given. We choose controls $u(\cdot)$ in the class $L^\infty(\Omega)$ and look for the state $y(\cdot)$ in the class $H_0^\alpha(\overline{\Omega}) \cap W^{1,2}(\Omega)$, where $\alpha\in(0,1)$. We assume that $\varphi(x) > 0 \forall x \in \overline{\Omega}$

Let us explain the introduced notation. $L^p(\Omega)$ is the space of all functions $y(\cdot)$, summable with degree $p\in[1,\infty)$, defined on the set Ω and having finite norm $|y(\cdot)|_p := \sqrt[p]{\int_\Omega |y(\cdot)|^p dx} < \infty, p \in [1,\infty)$, $|y(\cdot)|_\infty :=$ esssup $|y(x)|$.

$H_0^\alpha(\Omega)$ is the Banach space of continuous functions $y(\cdot)$, defined on Ω, vanished on the boundary ∂Ω of the set Ω and having finite norm

$$|y(\cdot)|_\Omega^{(\alpha)} := \max_\Omega |y| + < y(\cdot) >_{x,\Omega}^{(\alpha)}$$

where $< y(\cdot) >_{x,\Omega}^{(\gamma)} := \sup_{x',x''\in\Omega} \frac{|y(x'')-y(x')|}{|x''-x'|^\gamma} 0 < \gamma < 1$.

Let us now write down for problem (7), (8) the Pontryagin's maximum principle formulated in Theorem 1.

Let $[y(\cdot),u(\cdot)]$ be an optimal process in problem (7), (8). By Theorem 1 there is a function $\psi(\cdot)\in W_0^{1,\sigma}(\Omega)$, a number $v\ge 0$ and a finite regular Borel measure $(dx) \ge 0 (x \in \overline{\Omega})$ are such that:

$$u(x) = \arg\max_{\omega\in[u_-,u_+]} \psi(x) \left[wy(x)^3 - kw^2\right] \text{ a.a. } x \in \Omega \quad (9)$$

$$\mu(dx) \ge 0, \operatorname{supp} \mu(dx) \subset \{x \in \bar{\Omega} : y(x) = \varphi(x)\}, v \ge 0 \quad (10)$$

$$v + \mu(\overline{\Omega}) + \int_\Omega |\psi(x)| dx > 0 \quad (11)$$

$$\int_\Omega \psi(x) \left[\Delta h(x) - 3u(x)y^2(x)h(x)\right] dx + \int_\Omega h(x)\mu(dx) + 2v \int_\Omega h(x)y(x)dx = 0 \quad (12)$$

Equality (12) holds for any function $h(\cdot)\in H_0^\alpha(\Omega)\cap W^{1,2}(\Omega)$ for which $\Delta h(\cdot)\in L^\infty(\Omega)$. The notation $z^0=\arg\max_{z\in Z} f(x)$ means that the function $f(z)$ reaches its maximum on the set Z at the point z^0.

Lemma 1.

The inequality $y(x)\ge 0$ is valid $\forall x \in \overline{\Omega}$.

Proof of Lemma 1.

By the definition of generalized solution of the homogeneous Dirichlet problem [11] for the equation from (7) we have:

$$-\int_\Omega < \nabla y, \nabla h > dx = \int_\Omega u(x)y(x)^3 h(x)dx - k\int_\Omega u(x)^2 h(x)dx \quad (13)$$

The equality (13) holds for any function $h(\cdot)\in W_0^{1,2}(\Omega)$. According to Lemma 12 ([12, II, 3]) $h(\cdot) \coloneqq y\_(x)\in W\_{0}^{1,2}(\Omega)$, where $y\_(x) \coloneqq \min\{y(x),0\}$. Let $\Omega\_ \coloneqq \{x\in\Omega: y(x)<0\}$ and $\Omega^+ \coloneqq \Omega\backslash\Omega\_ = \{x\in\Omega: y(x)\ge 0\}$. It is easy to verify that $y\_(x)=y(x)$, $\nabla y\_(x)=\nabla y(x)$ $a.a. x\in\Omega\_$, $y\_(x)=0, \nabla y\_(x)=0$ $a.a. x\in\Omega^+$.

Substituting $h(\cdot) \coloneqq y\_(\cdot)$ into (13), we get

$$-\int_\Omega |\nabla y_-(x)|^2 dx = \int_\Omega u(x)y_-(x)^4 dx - k\int_\Omega u(x)^2 y_-(x)dx$$

Since $u(x)\ge u\_>0$ and $y\_(x)\le 0$, then both terms on the right side are non-negative, while the expression on the left side is not positive. Therefore, $\int_\Omega |\nabla y\_(x)|^2 dx=0$, and since $y\_(\cdot)\in W_0^{1,2}(\Omega)$, then $y\_(x)=0$ and that means $y(x)\ge 0$ for almost all x. Recalling that $y(\cdot)\in H_0^\alpha(\Omega)$, we come to the conclusion of the Lemma 1.

Remark 1.

Note that $y(x)>0$ at least at one point $x\in\Omega$. Indeed, otherwise $y(\cdot)\equiv 0$ and according to (7) $0=u(x)\ge u\_$, that is impossible due to inequality $u\_>0$.

The following auxiliary fact can be proved in a similar way.

Lemma 2.

Let $a(\cdot),f(\cdot)\in L^\infty(\Omega)$ and $h(\cdot)\in W_0^{1,2}(\Omega)$ be the solution to the Dirichlet problem:

$$\Delta h(x) = a(x)h(x) + f(x), h|_\Gamma \equiv 0 \quad (14)$$

If $a(x)\ge 0$ and $f(x)\le 0$ for almost all $x\in\Omega$, then $h(x) \ge 0 \forall x \in \overline{\Omega}$.

Remark 2.

By the theorem 14.1 ([12, III, 14]) $h(\cdot) \in H_0^\alpha(\overline{\Omega})$.

Lemma 2 allows for the following clarification.

Lemma 3.

Let, under the conditions of Lemma $2f(x)<0$ for almost all $x\in\Omega$.

Then $h(x)>0$ $\forall x\in int\Omega$.

Proof of Lemma 3.

The statement of Lemma 3 obviously follows from Lemma 2.

Let us establish its important consequence concerning the function $\psi(\cdot)$ from the maximum principle (9) – (12).

Lemma 4.

The function $\psi(x)>0$ for almost all $x\in\Omega$.

Proof of Lemma 4.

Consider a function $f(\cdot)\in L^\infty(\Omega)$ such that $f(x)\le 0$ for almost all $x\in\Omega$ and $f(x)<0$ for all points x from some set of positive measure. Let us define $h(\cdot)\in W_0^{1,2}(\Omega)$ as a generalized solution to the Dirichlet problem (14) with $a(x):=3u(x)y^2(x)$. Since $a(x)\ge u\_ \cdot y^2(x)\ge 0$ due to the second relation from (7), this solution exists and is uniquely determined. According to the Remark $2h(\cdot)\in H_0^\alpha(\overline{\Omega})$, whence in view of (14) $\Delta h(\cdot)\in L^\infty$. This means that the function $h(\cdot)$ can be substituted into (12)

$$\int_\Omega \psi(x)f(x)dx = -\int_\Omega h(x)\mu(dx) - 2v\int_\Omega h(x)y(x)dx \qquad (15)$$

It follows that the nondegeneracy condition (11) can be refined as follows:

$$v+\mu(\overline{\Omega}) > 0 \qquad (16)$$

Indeed, if (16) is violated, then according to (10) and (11) $v=0$ and $\mu(dx)=0$. But then (15) takes the form $\int_\Omega \psi(x)f(x)dx=0$, where the non-positive and non-zero function $f(\cdot)\in L^\infty(\Omega)$ is arbitrary. But then $\psi(\cdot)=0$, which, along with the equalities $v=0$ and $\mu(dx)=0$, contradicts (11). In (15) $h(x)>0\ \forall x\in int\Omega$ by Lemma 3. Since $\varphi(x)>0\forall x\in\overline{\Omega}$ by assumption, and $y(x)=0$ for $x\in\Gamma$ due to the boundary condition from (7), then according to (10) $supp\ \mu(dx)\subset int\ \Omega$ and $\mu(dx)\ge 0$. At the same time, by Lemma 1 and Remark 1, $y(x)\ge 0\forall x\in\overline{\Omega}$ and $max_{x\in\Omega}\ y(x)>0$. From this and from (16) it follows that in (15) the right-hand side is strictly negative and therefore $\int_\Omega \psi(x)f(x)dx<0$. Since here $f(\cdot)\in L^\infty(\Omega)$ is an arbitrary non-positive and non-zero function, we come to the conclusion of Lemma 4. Taking Lemma 4 into account, relation (9) is transformed to the form $u^0=\arg\ \max_{u\in[u_-,u_+]}\left[uy(x)^3-ku^2\right]$.

It follows that

$$u^0(x)=\chi\left\{\frac{\left[y^0(x)\right]^3}{2k}\right\}$$

where $\chi(v):=v$ for $u\_\le v\le u_+$, $\chi(v):=u\_$ for $v<u\_$, $\chi(v):=u_+$ for $v>u_+$.

Thus, the optimal process is generated by a nonlinear controller

$$u(x)=\chi\left\{\frac{y^3}{2k}\right\} \qquad (17)$$

Substituting this equality into the first equation from (7), taking (8) into account, leads to the relations

$$\Delta y - a(y)=0,\ y|_\Gamma\equiv 0 \qquad (18)$$

Here

$$a(y):=\chi\left\{\frac{y^3}{2k}\right\}y^3-k\chi^2\left\{\frac{y^3}{2k}\right\} \qquad (19)$$

Note that formulas (17), (18), (19) uniquely determine the process $[y(\cdot),u(\cdot)]\in\left(W_0^{1,2}\cap L^4\right)\times L^\infty$ which is optimal. Indeed, for this it is enough to verify that the boundary value problem (18) is uniquely solvable. For this purpose, we note that function (19) is strictly monotonic:

$$[a(y)-a(z)](y-z)>0\forall y\ne z$$

Therefore, the first equation from (18) satisfies the condition of strict monotonicity (9.33) ([12, IV, 9]):

$$\int_\Omega\left\{|\nabla y(x)-\nabla z(x)|^2+[a(y)-a(z)](y-z)\right\}dx>0$$

for any two elements y and z from $W_0^{1,2}(\Omega)\cap L^4(\Omega)$ that are not identically equal to each other. In addition, the coercivity condition (9.2) ([12, IV, 9]) is satisfied. Indeed,

$$\int_\Omega|\nabla y(x)|^2dx+\int_\Omega a(x)y(x)dx=\int_\Omega|\nabla y(x)|^2dx+\int_\Omega\chi\left\{\frac{y^3}{2k}\right\}y^4(x)dx-$$

$$k\int_\Omega\chi^2\left\{\frac{y^3}{2k}\right\}y(x)dx\ge\int_\Omega|\nabla y(x)|^2dx+u_-\int_\Omega y^4dx-ku_+\int_\Omega|y|dx$$

To estimate $\int_\Omega|y|dx$ let's use Young's inequality: $ab\le\frac{1}{p}(\varepsilon a)^p+\frac{1}{q}\left(\frac{b}{\varepsilon}\right)^q$ ([12, II, 1]), where $p>1$ and $q=\frac{p}{p-1}$.

Let us take here $a:=|y(x)|$, $b:=1$, $p:=4$, then $q:=4/3$. We get

$$\int_\Omega|y(x)|dx\le\frac{\varepsilon^4}{4}\int_\Omega|y(x)|^4dx+\frac{3}{4\varepsilon^{4/3}}\,\mathrm{mes}(\Omega),$$

Thus,

$$\int_\Omega|\nabla y(x)|^2dx+\int_\Omega a(x)y(x)dx\ge\int_\Omega|\nabla y(x)|^2dx+u_-\int_\Omega y^4dx-$$

$$-ku_+\frac{\varepsilon^4}{4}\int_\Omega y^4dx-\frac{3ku_+}{4\varepsilon^{4/3}}\,\mathrm{mes}(\Omega)$$

Choosing now

$$\varepsilon<\sqrt[4]{4\frac{u_-}{ku_+}}$$

we get an estimate of the form

$$\int_\Omega|\nabla y(x)|^2dx+\int_\Omega a(x)y(x)dx\ge\int_\Omega|\nabla y(x)|^2dx+x\int_\Omega y^4dx-C$$

where $\varkappa>0$, $C>0$.

This estimate means that the coercivity condition (9.2) ([12, IV, 9]) is actually true for *m=2* and *q=4*. The validity of condition (1) ([12, IV, 9]) is obvious. In view of the established facts, the existence of a solution to problem (18) is guaranteed by Theorem 9.1 ([12, IV, 9]), and the uniqueness is guaranteed by the concluding remark 9.1 ([12, IV, 9]).

Conclusion

The purpose of the paper was to demonstrate the practical application of the theory developed in [2] to the state-constrained optimal control problem for a Laplace equation. As a result, it was found that the optimal process in this problem is generated by a nonlinear optimal controller and its equation was obtained.

Конфликт интересов

Не указан.

Рецензия

Все статьи проходят рецензирование. Но рецензент или автор статьи предпочли не публиковать рецензию к этой статье в открытом доступе. Рецензия может быть предоставлена компетентным органам по запросу.

Conflict of Interest

None declared.

Review

All articles are peer-reviewed. But the reviewer or the author of the article chose not to publish a review of this article in the public domain. The review can be provided to the competent authorities upon request.

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